The Complex Plane
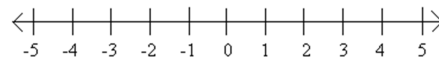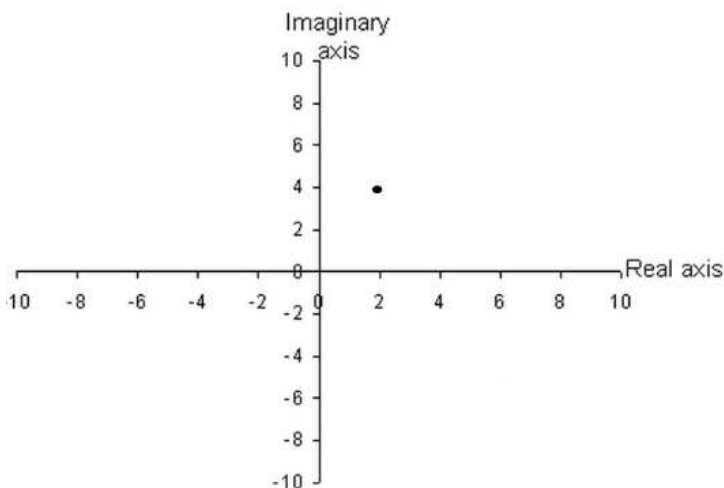
For real numbers, we are familar with graphing them on a number line like this:

A complex number is represented as $a + bi$, where a and b are real coefficients and i is $\sqrt{-1}$; a is the real part and bi is the imaginary part. We can plot a complex number by using an x-y graph, using the horizontal axis for the real part and the vertical axis for the imaginary part, like this:

This is called the complex plane. For the point $2 + 4i$, we would graph it on the complex plane 2 units to the right of the origin 4 units up from the origin (see ●). In general, the complex number $a + bi$ would have the coordinates (a, b) on the coordinate plane.
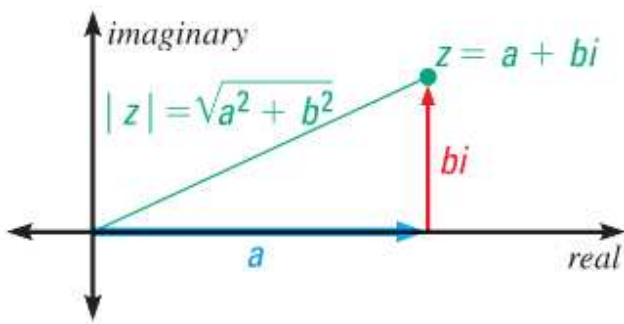

Another characteristic of each point on the complex plane is its absolute value, or modulus. If we say that if the complex number is z, so that $z = a + bi$, then the abolute value, or modulus of z is $|z| = \sqrt{a^2 + b^2}$. The distance formula applies here because the absolute value is the distance from zero and in this case zero is the origin. The Pythagorean theorem is then used to find the length of the hypotenuse of the right triangle equivalent to the modulus.

Example: Find the modulus of $3 + 4i$:

$$|z| = \sqrt{3^2 + 4^2}$$
$$= \sqrt{9 + 16}$$
$$= \sqrt{25}$$
$$= 5$$

Example: Plot the value $-2 - 3i$ on the complex plane:
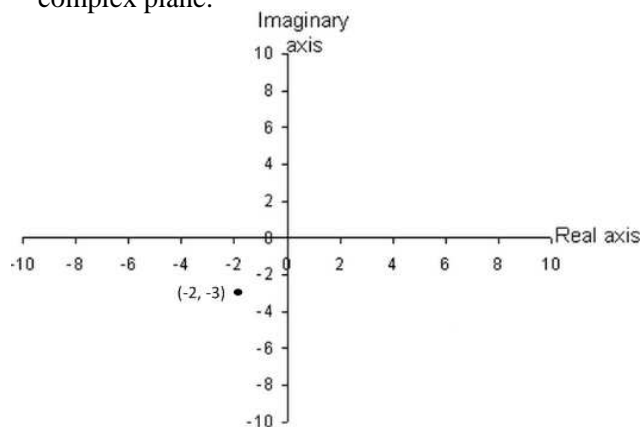
The Mandelbrot Set

Named for Benoit Mandelbrot, this is a particular set of complex numbers. To determine if a number is part of the Mandelbrot set, a sequence of numbers is generated using that number. The sequence generated for the complex number c is calculated from the recursive formula $z_{n+1} = (z_n)^2 + c$, with $z_0 = 0$. For any value c, the first few values of the sequence would be:

$$z_1 = (z_0)^2 + c \qquad z_2 = (z_1)^2 + c \qquad z_3 = (z_2)^2 + c$$
$$ = (0)^2 + c \qquad = (c)^2 + c \qquad = (c^2 + c)^2 + c \qquad z_4 = (z_3)^2 + c$$
$$ = c \qquad\quad = c^2 + c \qquad = (c^2 + c)^2 + c \qquad = ((c^2+c)^2 + c)^2 + c \qquad \text{and so on.}$$

If the absolute values in this sequence (the modulus of each z, or $|z|$) become infinitely large as the sequence continues, the sequence is considered unbounded and this point is not part of the Mandelbrot set. If the sequence of values does stay below some number, and does not increase to some infinitely large value, it is part of the Mandelbrot set. The following examples show numbers that are and are not part of the Mandelbrot set.

$$c = 1 + i$$

$$z_1 = (z_0)^2 + (1+i) \qquad z_2 = (z_1)^2 + c \qquad z_3 = (z_2)^2 + c \qquad z_4 = (z_3)^2 + c$$
$$ = (0)^2 + 1 + i \qquad\quad = (1+i)^2 + (1+i) \qquad = (1+3i)^2 + (1+i) \qquad = (-7+7i)^2 + (1+i)$$
$$ = 1 + i \qquad\qquad\quad = 1 + 2i - 1 + 1 + i \qquad = 1 + 6i - 9 + 1 + i \qquad = 49 - 98i - 49 + 1 + i$$
$$ \qquad\qquad\qquad\qquad = 1 + 3i \qquad\qquad\quad = -7 + 7i \qquad\qquad\quad = 1 - 97i$$
$$|z_1| = \sqrt{1^2 + 1^2} \qquad |z_2| = \sqrt{1^2 + 3^2} \qquad |z_3| = \sqrt{(-7)^2 + 7^2} \qquad |z_4| = \sqrt{1^2 + (-97)^2}$$
$$ = \sqrt{2} \qquad\qquad = \sqrt{10} \qquad\qquad = \sqrt{98} \qquad\qquad = \sqrt{9410}$$
$$ \approx 1.4 \qquad\qquad \approx 3.2 \qquad\qquad \approx 9.9 \qquad\qquad \approx 97.0$$

Since it looks like the sequence values are getting larger, $c = 1 + i$ is not in the Mandelbrot set.

$$c = -0.5 + 0i \qquad\qquad \text{since } b = 0, \text{ we can also use } c = -0.5$$

$$z_1 = (z_0)^2 + (-0.5 + 0i) \qquad z_2 = (z_1)^2 + c \qquad z_3 = (z_2)^2 + c \qquad z_4 = (z_3)^2 + c$$
$$ = (0)^2 - 0.5 + 0i \qquad\quad = (0.5)^2 + (-0.5) \qquad = (0.25)^2 + (-0.5) \qquad = (-0.4375)^2 + (-0.5)$$
$$ = -0.5 + 0i \qquad\qquad\quad = 0.25 - 0.5 \qquad\qquad = 0.0625 - 0.5 \qquad\quad = 0.1914 - 0.5$$
$$ \qquad\qquad\qquad\qquad = -0.25 \qquad\qquad\quad = -0.4375 \qquad\qquad = -0.3086$$
$$|z_1| = \sqrt{(-0.5)^2 + 0^2} \qquad |z_2| = \sqrt{(-0.25)^2 + 0^2} \qquad |z_3| = \sqrt{(-0.4375)^2 + 0^2} \qquad |z_4| = \sqrt{(-0.3086)^2 + 0^2}$$
$$ = \sqrt{0.25} \qquad\qquad = 0.25 \qquad\qquad = 0.4375 \qquad\qquad = 0.3086$$
$$ = 0.5$$

Although more iterations are needed to confirm that this point is in the Mandelbrot set, you can see that instead of getting larger, the values of the sequence are going up and down. This point happens to be in the Mandelbrot set.

Try developing this sequence for another complex number, more like the first example. Also try an example that is purely imaginary, like $c = 3i$.

The points that turn out to be in the Mandelbrot Set have an unusual pattern, which is shown below:
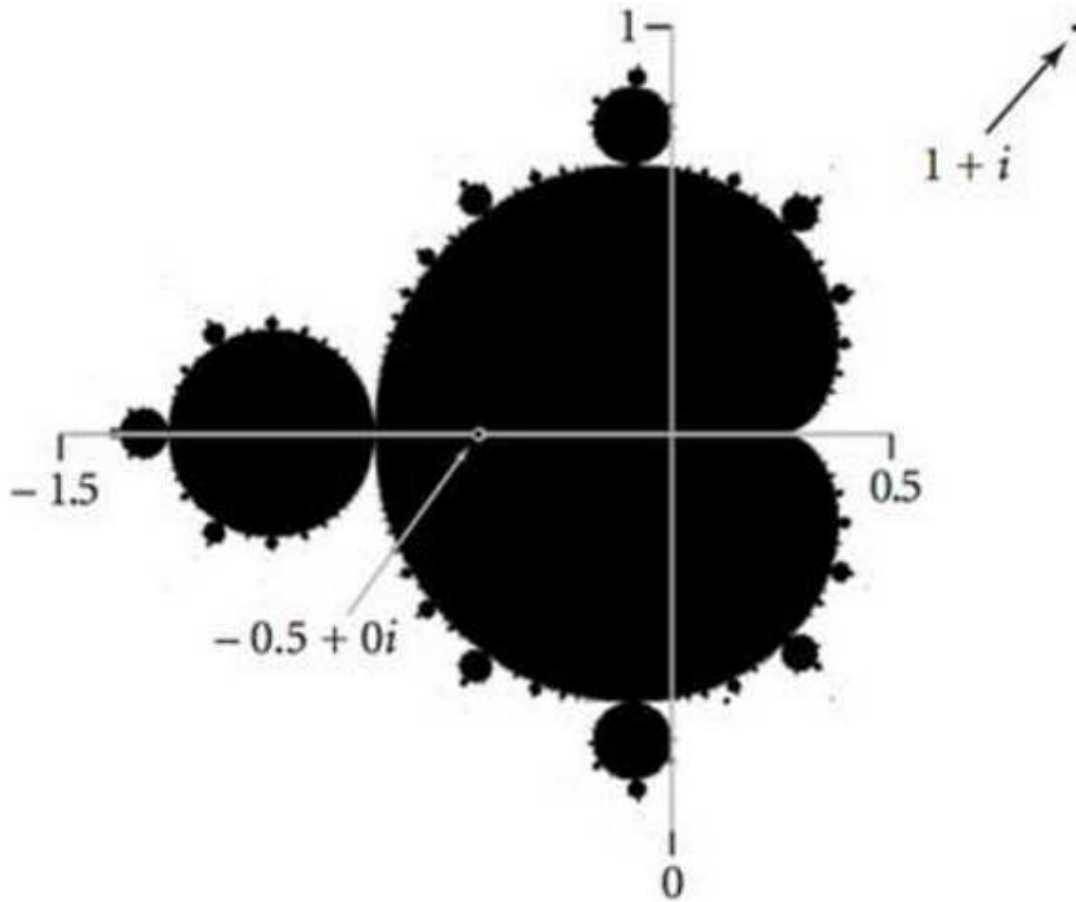
In this rendering, points in the Mandelbrot Set are shown in black. A striking feature of this set is that the "rough edges" do not become smooth as you zoom in. In fact, they maintain the same amount of detail no matter how far in you go. This self-similarity is a feature of fractal geometry, which is beyond what we are going to study this year. However, it is all based on the same calculations related to finding non-real roots of polynomials.
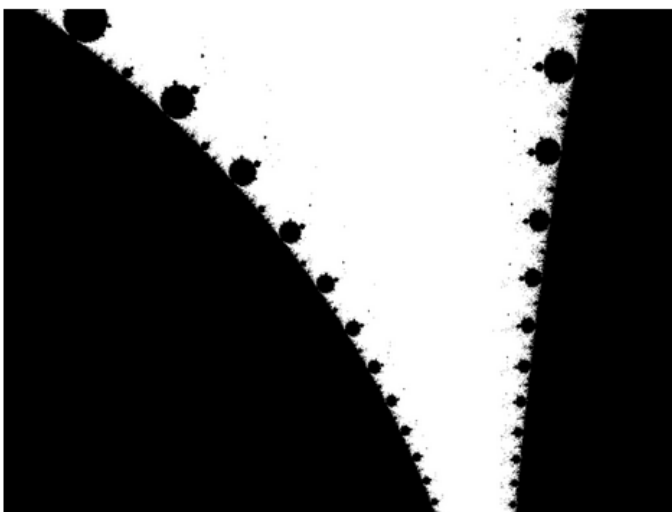
"Seahorse Valley"
Look at the interval from -1.0 to -0.5 in the figure above.

In other versions of this graph, different colors are used to show how fast different points become unbounded, which allows for the creation of even more interesting patterns.